

Buffon's problem with a star of needles and a lattice of parallelograms

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Abstract

A star of $n \geq 2$ line segments (needles) of equal length with common endpoint and constant angular spacing is randomly placed onto a lattice which is the union of two families of equidistant lines in the plane with angle α between the nonparallel lines. For odd n , we calculate the probabilities of exactly i intersections between the star and the lattice (for even n , see [3]). Using a geometrical method, we derive the limit distribution function of the relative number of intersections as $n \rightarrow \infty$. This function is independent of α . We show that the relative numbers for each of the two families are asymptotically independent random variables.

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1 Introduction

We consider the random throw of a star $\mathcal{S}_{n,\ell}$ of line segments onto a plane ruled with two families \mathcal{R}_a and \mathcal{R}_b of parallel lines,

$$\begin{aligned}\mathcal{R}_a &:= \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha = ka, k \in \mathbb{Z}\}, \\ \mathcal{R}_b &:= \{(x, y) \in \mathbb{R}^2 \mid y = mb, m \in \mathbb{Z}\},\end{aligned}$$

where a and b are positive real constants, $\alpha \in \mathbb{R}$, $0 < \alpha \leq \pi/2$, and put $\mathcal{R}_{a,b,\alpha} := \mathcal{R}_a \cup \mathcal{R}_b$. We denote the parallelogram

$$\mathcal{F} := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq b, y \cot \alpha \leq x \leq a \csc \alpha + y \cot \alpha\}$$

shown in Fig. 1 the *fundamental cell* of $\mathcal{R}_{a,b,\alpha}$. The star $\mathcal{S}_{n,\ell}$ consists of n ($2 \leq n < \infty$) line segments (*needles*) of equal length ℓ with common endpoint and constant angular spacing $2\pi/n$ between neighbouring needles. (The convex hull of $\mathcal{S}_{n,\ell}$ is the regular n -gon with circumscribed circle of radius ℓ .)

The *random throw* $\mathcal{S}_{n,\ell}$ of onto $\mathcal{R}_{a,b,\alpha}$ is defined as follows: The coordinates x and y of the centre point of $\mathcal{S}_{n,\ell}$ are random variables uniformly

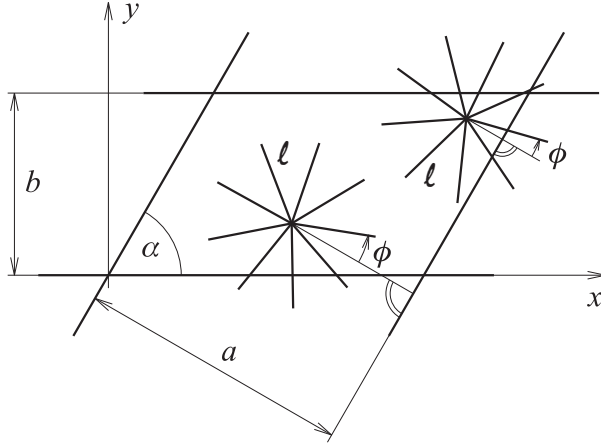


Figure 1: Star $\mathcal{S}_{n, \ell}$ (Example $n = 9$) and lattice and lattice $\mathcal{R}_{a, b, \alpha}$

distributed in $[y \cot \alpha, a \csc \alpha + y \cot \alpha]$ and $[0, b]$ resp.; the angle ϕ between the direction perpendicular to the lines of \mathcal{R}_a and a certain needle of $\mathcal{S}_{n, \ell}$ is a random variable uniformly distributed in $[0, 2\pi]$. All 3 random variables are stochastically independent. We assume $2\ell \sin(\frac{\pi}{n} \lfloor \frac{n}{2} \rfloor) \leq \min(a, b)$; in this case the probability that $\mathcal{S}_{n, \ell}$ intersects two lines of \mathcal{R}_a (or \mathcal{R}_b) at the same time is equal to zero. The maximum number M of intersections with \mathcal{R}_a (or \mathcal{R}_b) is then given by

$$M = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

In [8], Buffon published the solution of his famous needle problem. It is the calculation of the probability of the event that $\mathcal{S}_{2, \ell}$ intersects \mathcal{R}_a . ($\mathcal{S}_{2, \ell}$ can be considered as single needle of length 2ℓ .) Laplace [10, pp. 359-362] calculated the intersection probability for $\mathcal{S}_{2, \ell}$ and $\mathcal{R}_{a, b, \pi/2}$. Santaló [12] generalized this result for $\mathcal{R}_{a, b, \alpha}$, $0 < \alpha \leq \pi/2$, and derived the probabilities of 0, 1 or 2 intersection points (see also [13, p. 139]). Duma and Stoka [9] solved the problem for ellipses and $\mathcal{R}_{a, b, \pi/2}$. Ren and Zhang [11] and Aleman et al. [1] calculated the intersection probability for an arbitrary convex body K and $\mathcal{R}_{a, b, \alpha}$, and proved that for K there is a nonvanishing value of α for which the events K intersects \mathcal{R}_a and K intersects \mathcal{R}_b are independent; explicit results for regular n -gons ($n \geq 2$) and $\mathcal{R}_{a, b, \alpha}$ were obtained by B  sel [4]. In [3], B  sel calculated the probabilities of exactly i intersections for $\mathcal{S}_{n, \ell}$ with even $n \geq 2$ and $\mathcal{R}_{a, b, \alpha}$. Bonanzinga [7] found the intersection probabilities for $\mathcal{S}_{3, \ell}$ and $\mathcal{R}_{a, b, \alpha}$, $\pi/3 \leq \alpha \leq \pi/2$.

In the Sections 2 and 3 we calculate the probabilities of exactly i intersections for $\mathcal{S}_{n, \ell}$ with odd n , $3 \leq n < \infty$, and $\mathcal{R}_{a, b, \alpha}$, $0 < \alpha \leq \pi/2$. In Section 4 we investigate the distribution functions of the *relative number of intersections* for $n \in \mathbb{N}$, $n \geq 2$. Using a geometrical method, we derive the limit distribution as $n \rightarrow \infty$. For abbreviation we put $\lambda = \ell/a$, $\mu = \ell/b$, and $\lfloor \cdot \rfloor$ for the integer part of \cdot .

2 Intersection probabilities

Theorem 1. *The probabilities $p(i)$ of exactly i intersections between $\mathcal{S}_{n,\ell}$ and $\mathcal{R}_{a,b,\alpha}$ are for odd $n \geq 3$, $2 \max(\lambda, \mu) \sin(\frac{\pi}{n} \lfloor \frac{n}{2} \rfloor) \leq 1$ and $0 < \alpha \leq \frac{\pi}{2n}$ given by*

$$p(i) = \begin{cases} 1 - \left[\frac{2n(\lambda+\mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda\mu}{\pi} f_0(\alpha) \right], & \text{if } i = 0, \\ \frac{8n(\lambda+\mu)}{\pi} \sin^2 \frac{\pi}{2n} \sin \frac{i\pi}{n} - \frac{4n\lambda\mu}{\pi} \left[f_1(\alpha) \sin \frac{i\pi}{n} \right. \\ \quad \left. - f_4(\alpha) \left(\cot \frac{\pi}{n} \sin \frac{i\pi}{n} - i \cos \frac{i\pi}{n} \right) \right], & \text{if } 1 \leq i \leq M-2, \\ \frac{4n(\lambda+\mu)}{\pi} \left(\cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{2n\lambda\mu}{\pi} \left[f_2(\alpha) \right. \\ \quad \left. - 2f_4(\alpha) \left(\cot \frac{\pi}{n} \sin \frac{i\pi}{n} - i \cos \frac{i\pi}{n} \right) \right], & \text{if } i = M-1, \\ \frac{4n(\lambda+\mu)}{\pi} \sin^2 \frac{\pi}{4n} - \frac{n\lambda\mu}{2\pi} [4f_3(\alpha) - f_7(\alpha)], & \text{if } i = M \text{ and } n = 3, \\ \frac{4n(\lambda+\mu)}{\pi} \sin^2 \frac{\pi}{4n} - \frac{2n\lambda\mu}{\pi} \left[f_3(\alpha) - 4f_5(\alpha) \sin \frac{\pi}{n} \right. \\ \quad \left. - f_4(\alpha) \left\{ (n-5) \sin \frac{\pi}{2n} + 2 \csc \frac{\pi}{n} \cos \frac{5\pi}{2n} \right\} \right], & \text{if } i = M \text{ and } n \geq 5, \\ \frac{4n\lambda\mu}{\pi} \left[2f_6(\alpha) \sin \frac{(i-M)\pi}{n} + 2f_5(\alpha) \sin \frac{(i+1-M)\pi}{n} \right. \\ \quad \left. - f_4(\alpha) \left\{ (2M-i-3) \cos \frac{i\pi}{n} \right. \right. \\ \quad \left. \left. - \csc \frac{\pi}{n} \sin \frac{(2M-i-3)\pi}{n} \right\} \right], & \text{if } M+1 \leq i \leq 2M-3, \\ \frac{n\lambda\mu}{2\pi} \left[16f_6(\alpha) \cos \frac{3\pi}{2n} + f_7(\alpha) \right], & \text{if } i = 2M-2 \text{ and } n \geq 5, \\ \frac{n\lambda\mu}{\pi} f_8(\alpha), & \text{if } i = 2M-1, \\ \frac{n\lambda\mu}{2\pi} f_9(\alpha), & \text{if } i = 2M, \end{cases}$$

where

$$\begin{aligned} f_0(\alpha) &= 2 \left[\frac{\pi}{n} \cos \alpha + g\left(\frac{\pi}{n} - \alpha\right) + h(\alpha) \right] \cos^2 \frac{\pi}{2n}, \\ f_1(\alpha) &= \left[\frac{\pi}{n} \cos \alpha + g\left(\frac{\pi}{n} - \alpha\right) + h(\alpha) \right] \sin \frac{\pi}{n}, \\ f_2(\alpha) &= \left[\frac{2\pi}{n} \cos \frac{\pi}{2n} \cos \alpha + g\left(\frac{3\pi}{2n} - \alpha\right) - g\left(\frac{\pi}{2n} - \alpha\right) + h\left(\frac{\pi}{2n} + \alpha\right) \right] \sin \frac{\pi}{n}, \\ f_3(\alpha) &= g\left(\frac{\pi}{2n} - \alpha\right) \sin \frac{\pi}{n}, \\ f_4(\alpha) &= \left[\frac{\pi}{n} \cos \alpha + g\left(\frac{\pi}{n} - \alpha\right) + h(\alpha) \right] \sin^2 \frac{\pi}{2n}, \\ f_5(\alpha) &= \left[\frac{2\pi}{n} \cos \frac{\pi}{2n} \cos \alpha + g\left(\frac{3\pi}{2n} - \alpha\right) - g\left(\frac{\pi}{2n} - \alpha\right) + h\left(\frac{\pi}{2n} + \alpha\right) \right] \sin^2 \frac{\pi}{2n}, \\ f_6(\alpha) &= g\left(\frac{\pi}{2n} - \alpha\right) \sin^2 \frac{\pi}{2n}, \\ f_7(\alpha) &= \frac{\pi}{n} \left(3 - 2 \cos \frac{2\pi}{n} \right) \cos \alpha - g\left(\frac{3\pi}{n} - \alpha\right) + 3g\left(\frac{2\pi}{n} - \alpha\right) - g\left(\frac{\pi}{n} - \alpha\right) \\ &\quad + 7h(\alpha) - h\left(\frac{\pi}{n} + \alpha\right) - h\left(\frac{2\pi}{n} + \alpha\right) \end{aligned}$$

$$f_8(\alpha) = -\frac{\pi}{n} \cos \alpha - g\left(\frac{2\pi}{n} - \alpha\right) + 2g\left(\frac{\pi}{n} - \alpha\right) - 4h(\alpha) + h\left(\frac{\pi}{n} + \alpha\right),$$

$$f_9(\alpha) = \frac{\pi}{n} \cos \alpha - g\left(\frac{\pi}{n} - \alpha\right) + 3h(\alpha)$$

with

$$g(x) = \sin x + \alpha \cos x \quad \text{and} \quad h(x) = \sin x - \alpha \cos x.$$

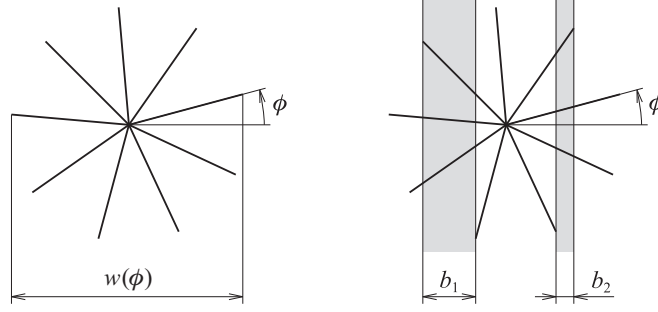


Figure 2: $w(\phi)$ and stripes of $s(3, \phi)$ for $\mathcal{S}_{9, \ell}$

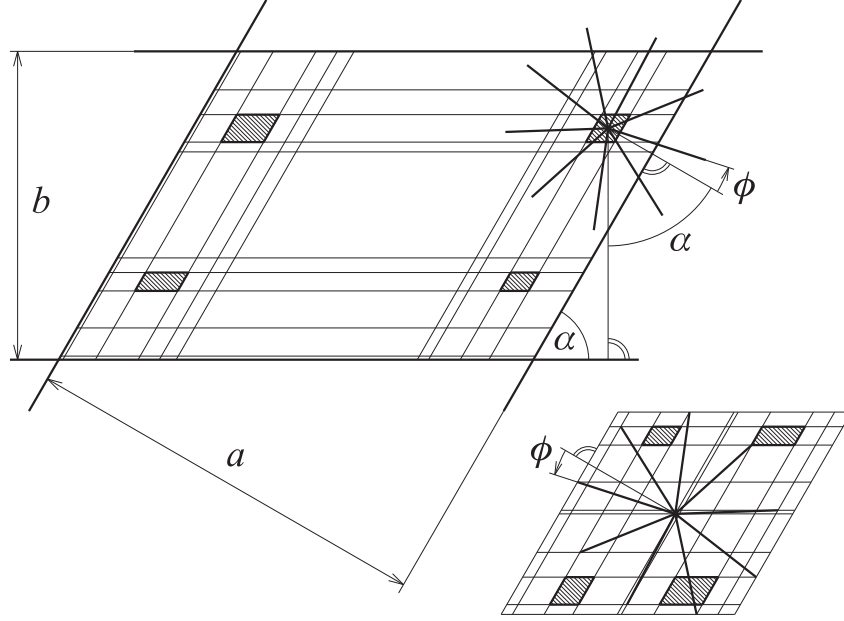


Figure 3: $E_{3,2}$ for star $\mathcal{S}_{9, \ell}$, lattice $\mathcal{R}_{a, b, \alpha}$ and fixed value of ϕ

Proof. We denote by $w(\phi)$ the width of $\mathcal{S}_{n, \ell}$ (with angle ϕ) perpendicular to the lines of \mathcal{R}_a (see Fig. 2 (left side) and Fig. 3), and by $s(k, \phi)$ the *breadth functions* of exactly k , $k \in \{1, 2, \dots, M\}$, intersections between $\mathcal{S}_{n, \ell}$ (with angle ϕ) and \mathcal{R}_a . $s(k, \phi)$ is the breadth of one stripe or the sum of the breadths of two stripes. An example of $s(3, \phi)$ for $\mathcal{S}_{9, \ell}$ is shown on the

right side of Fig. 2. Here, $s(3, \phi)$ is the sum of the breadths $b_1 = b_1(\phi)$ and $b_2 = b_2(\phi)$. From the symmetry of $\mathcal{S}_{n, \ell}$, it follows that w and $s(k, \cdot)$ are π/n -periodic functions. In the following, we have to consider these functions in the half-open intervals:

$$\mathcal{I}_1 := \left[0, \frac{\pi}{2n}\right), \quad \mathcal{I}_2 := \left[\frac{\pi}{2n}, \frac{\pi}{n}\right) \quad \text{and} \quad \mathcal{I}_3 := \left[\frac{\pi}{n}, \frac{3\pi}{2n}\right).$$

The required restrictions of the function w are given by

$$\begin{aligned} w_{12}(\phi) &:= w|_{\mathcal{I}_1 \cup \mathcal{I}_2}(\phi) = 2\ell \cos \frac{\pi}{2n} \cos\left(\phi - \frac{\pi}{2n}\right), \\ w_3(\phi) &:= w|_{\mathcal{I}_3}(\phi) = 2\ell \cos \frac{\pi}{2n} \cos\left(\phi - \frac{3\pi}{2n}\right). \end{aligned}$$

For $s(k, \cdot)$ and $1 \leq k \leq M-2$, one finds

$$\begin{aligned} s_{12}(k, \phi) &:= s|_{\mathcal{I}_1 \cup \mathcal{I}_2}(k, \phi) = 4\ell \sin \frac{k\pi}{n} \sin \frac{\pi}{2n} \cos\left(\phi - \frac{\pi}{2n}\right), \\ s_3(k, \phi) &:= s|_{\mathcal{I}_3}(k, \phi) = 4\ell \sin \frac{k\pi}{n} \sin \frac{\pi}{2n} \cos\left(\phi - \frac{3\pi}{2n}\right), \end{aligned}$$

for $k = M-1$,

$$\begin{aligned} s_1(k, \phi) &:= s|_{\mathcal{I}_1}(k, \phi) = \ell \left[2 \cos \frac{\pi}{2n} \sin \phi - \sin\left(\phi - \frac{3\pi}{2n}\right) \right], \\ s_2(k, \phi) &:= s|_{\mathcal{I}_2}(k, \phi) = \ell \left[-2 \cos \frac{\pi}{2n} \sin\left(\phi - \frac{\pi}{n}\right) + \sin\left(\phi + \frac{\pi}{2n}\right) \right], \\ s_3(k, \phi) &:= s|_{\mathcal{I}_3}(k, \phi) = \ell \left[2 \cos \frac{\pi}{2n} \sin\left(\phi - \frac{\pi}{n}\right) - \sin\left(\phi - \frac{5\pi}{2n}\right) \right], \end{aligned}$$

and for $k = M$,

$$\begin{aligned} s_1(k, \phi) &= s|_{\mathcal{I}_1}(k, \phi) = -\ell \sin\left(\phi - \frac{\pi}{2n}\right), \\ s_2(k, \phi) &= s|_{\mathcal{I}_2}(k, \phi) = \ell \sin\left(\phi - \frac{\pi}{2n}\right), \\ s_3(k, \phi) &= s|_{\mathcal{I}_3}(k, \phi) = -\ell \sin\left(\phi - \frac{3\pi}{2n}\right). \end{aligned}$$

$E_{k, m}$, $0 \leq k, m < M$, denotes the event that $\mathcal{S}_{n, \ell}$ has exactly k intersections with \mathcal{R}_a and (at the same time) exactly m intersections with \mathcal{R}_b . For fixed value of ϕ , this event occurs if the centre point of $\mathcal{S}_{n, \ell}$ is in one, two or four disjunct parallelograms that are subsets of \mathcal{F} . (An example is shown in Fig. 3. For the given angle ϕ , the event $E_{3, 2}$ occurs if the centre point of $\mathcal{S}_{9, \ell}$ is in one of the four hatched parallelograms.) $s(k, \phi) s(m, \phi + \alpha) / \sin \alpha$ is the area of the one parallelogram or the sum of the areas of the two or four parallelograms if $1 \leq k, m < M$. Therefore, the conditional probability of the event $E_{k, m}$ for fixed angle ϕ is given by

$$P(E_{k, m} | \phi) = \frac{s(k, \phi) s(m, \phi + \alpha) / \sin \alpha}{\text{Area } \mathcal{F}} = \frac{1}{ab} s(k, \phi) s(m, \phi + \alpha).$$

For $0 \leq k, m < M$, we have

$$P(E_{0, 0} | \phi) = \frac{1}{ab} [a - w(\phi)] [b - w(\phi + \alpha)],$$

$$P(E_{0,m} | \phi) = \frac{1}{ab} [a - w(\phi)] s(m, \phi + \alpha),$$

$$P(E_{k,0} | \phi) = \frac{1}{ab} s(k, \phi) [b - w(\phi + \alpha)].$$

The density function of the random variable ϕ is given by

$$f(\phi) = \begin{cases} \frac{n}{\pi} & \text{if } \phi \in \mathcal{I}_1 \cup \mathcal{I}_2, \\ 0 & \text{if } \phi \in \mathbb{R} \setminus \mathcal{I}_1 \cup \mathcal{I}_2. \end{cases}$$

Therefore, the (total) probability of the event $E_{k,m}$ is given by

$$P(E_{k,m}) = \int_0^{\pi/n} P(E_{k,m} | \phi) f(\phi) d\phi = \frac{n}{\pi} \int_0^{\pi/n} P(E_{k,m} | \phi) d\phi.$$

From the piecewise definition of the functions w and $s(k, \cdot)$, it follows that we have to distinguish (in general) the cases

$$0 \leq \phi < \frac{\pi}{2n} - \alpha, \quad \frac{\pi}{2n} - \alpha \leq \phi < \frac{\pi}{2n}, \quad \frac{\pi}{2n} \leq \phi < \frac{\pi}{n} - \alpha, \quad \frac{\pi}{n} - \alpha \leq \phi < \frac{\pi}{n}.$$

We calculate the probabilities $P(E_{k,m})$ in some examples. For $k = 0$ and $1 \leq m \leq M - 2$, we get

$$\begin{aligned} P(E_{0,m}) &= \frac{n}{\pi ab} \left(\int_0^{\pi/n-\alpha} + \int_{\pi/n-\alpha}^{\pi/n} \right) [a - w(\phi)] s(m, \phi + \alpha) d\phi \\ &= \frac{n}{\pi ab} \left(\int_0^{\pi/n-\alpha} [a - w_{12}(\phi)] s_{12}(m, \phi + \alpha) d\phi \right. \\ &\quad \left. + \int_{\pi/n-\alpha}^{\pi/n} [a - w_{12}(\phi)] s_3(m, \phi + \alpha) d\phi \right) \\ &= \frac{8n\mu}{\pi} \sin^2 \frac{\pi}{2n} \sin \frac{m\pi}{n} - \frac{2n\lambda\mu}{\pi} \sin \frac{m\pi}{n} f_1(\alpha). \end{aligned}$$

Due to symmetry, for $1 \leq k \leq M - 2$ and $m = 0$, we get

$$P(E_{k,0}) = \frac{8n\lambda}{\pi} \sin^2 \frac{\pi}{2n} \sin \frac{k\pi}{n} - \frac{2n\lambda\mu}{\pi} \sin \frac{k\pi}{n} f_1(\alpha).$$

For $1 \leq k, m \leq M - 2$, we find

$$\begin{aligned} P(E_{k,m}) &= \frac{n}{\pi ab} \left(\int_0^{\pi/n-\alpha} + \int_{\pi/n-\alpha}^{\pi/n} \right) s(k, \phi) s(m, \phi + \alpha) d\phi \\ &= \frac{8n\lambda\mu}{\pi} \sin \frac{k\pi}{n} \sin \frac{m\pi}{n} f_4(\alpha), \end{aligned}$$

for $k = M - 1$ and $m = M$,

$$P(E_{M-1,M}) = \frac{n}{\pi ab} \left(\int_0^{\frac{\pi}{2n}-\alpha} + \int_{\frac{\pi}{2n}-\alpha}^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\frac{\pi}{n}-\alpha} + \int_{\frac{\pi}{n}-\alpha}^{\frac{\pi}{n}} \right) s(M-1, \phi) \\ \times s(M, \phi + \alpha) d\phi = \frac{n\lambda\mu}{2\pi} f_8(\alpha),$$

and due to symmetry, $P(E_{M,M-1}) = P(E_{M-1,M})$.

The remaining calculations deliver the results

$$\begin{aligned} P(E_{k,m}) &= 1 - \left(\frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda\mu}{\pi} f_0(\alpha) \right), \quad (k = 0 = m), \\ P(E_{k,m}) &= \frac{4n\mu}{\pi} \left(\cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{n\lambda\mu}{\pi} f_2(\alpha), \quad (k = 0, m = M - 1), \\ P(E_{k,m}) &= \frac{4n\lambda}{\pi} \left(\cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{n\lambda\mu}{\pi} f_2(\alpha), \quad (k = M - 1, m = 0), \\ P(E_{k,m}) &= \frac{4n\mu}{\pi} \sin^2 \frac{\pi}{4n} - \frac{n\lambda\mu}{\pi} f_3(\alpha), \quad (k = 0, m = M), \\ P(E_{k,m}) &= \frac{4n\lambda}{\pi} \sin^2 \frac{\pi}{4n} - \frac{n\lambda\mu}{\pi} f_3(\alpha), \quad (k = M, m = 0), \\ P(E_{k,m}) &= \frac{4n\lambda\mu}{\pi} \sin \frac{k\pi}{n} f_5(\alpha), \quad (1 \leq k \leq M - 2, m = M - 1), \\ P(E_{k,m}) &= \frac{4n\lambda\mu}{\pi} \sin \frac{m\pi}{n} f_5(\alpha), \quad (k = M - 1, 1 \leq m \leq M - 2), \\ P(E_{k,m}) &= \frac{4n\lambda\mu}{\pi} \sin \frac{k\pi}{n} f_6(\alpha), \quad (1 \leq k \leq M - 2, m = M), \\ P(E_{k,m}) &= \frac{4n\lambda\mu}{\pi} \sin \frac{m\pi}{n} f_6(\alpha), \quad (k = M, 1 \leq m \leq M - 2), \\ P(E_{k,m}) &= \frac{n\lambda\mu}{2\pi} f_7(\alpha), \quad (k = M - 1 = m), \\ P(E_{k,m}) &= \frac{n\lambda\mu}{2\pi} f_9(\alpha), \quad (k = M = m). \end{aligned}$$

The probabilities $p(i)$ of exactly i intersections between $\mathcal{S}_{n,\ell}$ und $\mathcal{R}_{a,b,\alpha}$ are given by

$$p(i) = \begin{cases} \sum_{k=0}^i P(E_{k,i-k}) & \text{for } 0 \leq i \leq M, \\ \sum_{k=i-M}^M P(E_{k,i-k}) & \text{for } M + 1 \leq i \leq 2M. \end{cases}$$

We have

$$p(0) = P(E_{0,0}) = 1 - \left(\frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda\mu}{\pi} f_0(\alpha) \right).$$

For $1 \leq i \leq M-2$, one finds

$$\begin{aligned} p(i) &= P(E_{0,i}) + P(E_{i,0}) + \sum_{k=1}^{i-1} P(E_{k,i-k}) \\ &= \frac{8n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{2n} \sin \frac{i\pi}{n} - \frac{4n\lambda\mu}{\pi} f_1(\alpha) \sin \frac{i\pi}{n} \\ &\quad + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=1}^{i-1} \sin \frac{k\pi}{n} \sin \frac{(i-k)\pi}{n} \end{aligned}$$

with

$$\sum_{k=1}^{i-1} \sin \frac{k\pi}{n} \sin \frac{(i-k)\pi}{n} = \frac{1}{2} \left(\cot \frac{\pi}{n} \sin \frac{i\pi}{n} - i \cos \frac{i\pi}{n} \right).$$

For $i = M-1$, we get

$$\begin{aligned} p(i) &= P(E_{0,M-1}) + P(E_{M-1,0}) + \sum_{k=1}^{M-2} P(E_{k,i-k}) \\ &= \frac{4n(\lambda + \mu)}{\pi} \left(\cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{2n\lambda\mu}{\pi} f_2(\alpha) \\ &\quad + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=1}^{i-1} \sin \frac{k\pi}{n} \sin \frac{(i-k)\pi}{n} \end{aligned}$$

with the sum as above. For $i = M = 2$ and $n = 3$, we find

$$\begin{aligned} p(M) &= P(E_{0,M}) + P(E_{M,0}) + P(E_{M-1,M-1}) \\ &= \frac{4n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{4n} - \frac{2n\lambda\mu}{\pi} f_3(\alpha) + \frac{n\lambda\mu}{2\pi} f_7(\alpha) \sin \frac{\pi}{n}, \end{aligned}$$

and for $i = M$ and $n \geq 5$,

$$\begin{aligned} p(M) &= P(E_{0,M}) + P(E_{M,0}) + P(E_{1,M-1}) + P(E_{M-1,1}) + \sum_{k=2}^{M-2} P(E_{k,i-k}) \\ &= \frac{4n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{4n} - \frac{2n\lambda\mu}{\pi} f_3(\alpha) + \frac{8n\lambda\mu}{\pi} f_5(\alpha) \sin \frac{\pi}{n} \\ &\quad + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=2}^{M-2} \sin \frac{k\pi}{n} \sin \frac{(M-k)\pi}{n} \end{aligned}$$

with

$$\begin{aligned} \sum_{k=2}^{M-2} \sin \frac{k\pi}{n} \sin \frac{(M-k)\pi}{n} &= \frac{1}{2} \left(-(M-3) \cos \frac{M\pi}{n} + \csc \frac{\pi}{n} \sin \frac{(M-3)\pi}{n} \right) \\ &= \frac{1}{4} \left((n-5) \sin \frac{\pi}{2n} + 2 \csc \frac{\pi}{n} \cos \frac{5\pi}{2n} \right). \end{aligned}$$

For the case $M + 1 \leq i \leq 2M - 3$, we put $i = 2M - \nu$. So we have to consider all ν with $3 \leq \nu \leq M - 1$. One finds

$$\begin{aligned}
p(2M - \nu) &= \sum_{k=(2M-\nu)-M}^M P(E_{k, 2M-\nu-k}) = \sum_{k=M-\nu}^M P(E_{k, 2M-\nu-k}) \\
&= P(E_{M-\nu, M}) + P(E_{M, M-\nu}) + P(E_{M-(\nu-1), M-1}) \\
&\quad + P(E_{M-1, M-(\nu-1)}) + \sum_{k=M-(\nu-2)}^{M-2} P(E_{k, 2M-\nu-k}) \\
&= \frac{8n\lambda\mu}{\pi} f_6(\alpha) \sin \frac{(M-\nu)\pi}{n} + \frac{8n\lambda\mu}{\pi} f_5(\alpha) \sin \frac{[M-(\nu-1)]\pi}{n} \\
&\quad + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=M-(\nu-2)}^{M-2} \sin \frac{k\pi}{n} \sin \frac{(2M-\nu-k)\pi}{n}
\end{aligned}$$

with

$$\begin{aligned}
&\sum_{k=M-(\nu-2)}^{M-2} \sin \frac{k\pi}{n} \sin \frac{(2M-\nu-k)\pi}{n} \\
&= \frac{1}{2} \left(-(\nu-3) \cos \frac{(2M-\nu)\pi}{n} + \csc \frac{\pi}{n} \sin \frac{(\nu-3)\pi}{n} \right),
\end{aligned}$$

and therefore, with $\nu = 2M - i$,

$$\begin{aligned}
p(i) &= \frac{4n\lambda\mu}{\pi} \left[2f_6(\alpha) \sin \frac{(i-M)\pi}{n} + 2f_5(\alpha) \sin \frac{(i+1-M)\pi}{n} \right. \\
&\quad \left. - f_4(\alpha) \left((2M-i-3) \cos \frac{i\pi}{n} - \csc \frac{\pi}{n} \sin \frac{(2M-i-3)\pi}{n} \right) \right].
\end{aligned}$$

For $i = 2M - 2$ and $n \geq 5$, we get

$$\begin{aligned}
p(2M - 2) &= \sum_{k=M-2}^M P(E_{k, (2M-2)-k}) \\
&= P(E_{M-2, M}) + P(E_{M, M-2}) + P(E_{M-1, M-1}) \\
&= \frac{8n\lambda\mu}{\pi} f_6(\alpha) \sin \frac{(M-2)\pi}{n} + \frac{n\lambda\mu}{2\pi} f_7(\alpha) \\
&= \frac{n\lambda\mu}{2\pi} \left(16f_6(\alpha) \cos \frac{3\pi}{2n} + f_7(\alpha) \right).
\end{aligned}$$

Furthermore, we find

$$\begin{aligned}
p(2M - 1) &= \sum_{k=M-1}^M P(E_{k, (2M-1)-k}) = P(E_{M-1, M}) + P(E_{M, M-1}) \\
&= \frac{n\lambda\mu}{\pi} f_8(\alpha)
\end{aligned}$$

and finally

$$p(2M) = P(E_{M,M}) = \frac{n\lambda\mu}{2\pi} f_9(\alpha).$$

So, the proof is complete. \square

In the following, we write $p(i, \alpha)$ instead of $p(i)$ and $P_\alpha(E_{k,m})$ instead of $P(E_{k,m})$ to emphasize the dependence on α .

Theorem 2. *For fixed values of odd $n \geq 3$, a , b and ℓ , the function*

$$p(i, \cdot) : [0, \pi/2] \rightarrow [0, 1], \quad \alpha \mapsto p(i, \alpha)$$

is π/n -periodic. The restriction $p|_{[0, \pi/n]}$ is symmetric in relation to the line $\alpha = \pi/(2n)$.

Proof. The functions w and $s(k, \cdot)$, $1 \leq k \leq M$, are π/n -periodic. For $1 \leq k, m \leq M$ and $\nu \in \mathbb{Z}$ we get

$$\begin{aligned} P_{\alpha + \nu\pi/n}(E_{k,m}) &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi) s\left(m, \phi + \alpha + \nu \frac{\pi}{n}\right) d\phi \\ &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi) s(m, \phi + \alpha) d\phi = P_\alpha(E_{k,m}). \end{aligned}$$

This result holds for all values of k and m , $0 \leq k, m \leq M$. Since $p(i, \alpha)$ is a sum of π/n -periodic functions, it is π/n -periodic.

$s(k, \phi) s(m, \phi + \alpha)$ are π/n -periodic functions. Hence

$$\begin{aligned} P_\alpha(E_{k,m}) &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi) s(m, \phi + \alpha) d\phi \\ &= \frac{n}{\pi ab} \int_{-\alpha}^{\pi/n - \alpha} s(k, \phi) s(m, \phi + \alpha) d\phi \\ &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, u - \alpha) s(m, u) du, \end{aligned}$$

and therefore, with $\nu \in \mathbb{Z}$,

$$\begin{aligned} P_{\nu\pi/n - \alpha}(E_{k,m}) &= \frac{n}{\pi ab} \int_0^{\pi/n} s\left(k, u - \left(\nu \frac{\pi}{n} - \alpha\right)\right) s(m, u) du \\ &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, u + \alpha) s(m, u) du = P_\alpha(E_{m,k}), \end{aligned}$$

For $1 \leq k, m \leq M$, it follows that

$$\begin{aligned} P_{\nu\pi/n - \alpha}(E_{k,k}) &= P_\alpha(E_{k,k}), \\ P_{\nu\pi/n - \alpha}(E_{k,m}) + P_{\nu\pi/n - \alpha}(E_{m,k}) &= P_\alpha(E_{k,m}) + P_\alpha(E_{m,k}). \end{aligned}$$

Analogously, one gets

$$\begin{aligned} P_{\nu\pi/n-\alpha}(E_{0,0}) &= P_{\alpha}(E_{0,0}), \\ P_{\nu\pi/n-\alpha}(E_{0,m}) + P_{\nu\pi/n-\alpha}(E_{m,0}) &= P_{\alpha}(E_{0,m}) + P_{\alpha}(E_{m,0}). \end{aligned}$$

With $\nu = 1$, we have

$$\begin{aligned} p(0, \pi/n - \alpha) &= P_{\pi/n-\alpha}(E_{0,0}) = P_{\alpha}(E_{0,0}) = p(0, \alpha), \\ p(2M, \pi/n - \alpha) &= P_{\pi/n-\alpha}(E_{M,M}) = P_{\alpha}(E_{M,M}) = p(2M, \alpha). \end{aligned}$$

For $1 \leq i \leq 2M - 1$, we find: If i is odd, $p(i, \alpha)$ is the sum of terms $P_{\alpha}(E_{k,i-k}) + P_{\alpha}(E_{i-k,k})$. If i is even, $p(i, \alpha)$ is the sum of terms $P_{\alpha}(E_{k,i-k}) + P_{\alpha}(E_{i-k,k})$ and one term $P(E_{i/2,i/2})$.

So, for every i , $0 \leq i \leq 2M$, we have $p(i, \pi/n - \alpha) = p(i, \alpha)$; therefore, the restriction $p(i, \alpha)|_{[0, \pi/n]}$ is symmetric in relation to the line $\alpha = \pi/(2n)$. \square

From Theorem 2 one easily gets the following corollary:

Corollary 1. *The probabilities $p(i, \alpha)$ for $0 < \alpha \leq \pi/2$ are given by*

$$p(i, \alpha) = \begin{cases} p(i, \alpha - \delta(\alpha)) & \text{if } \alpha - \delta(\alpha) \leq \frac{\pi}{2n}, \\ p\left(i, \frac{\pi}{n} - [\alpha - \delta(\alpha)]\right) & \text{if } \alpha - \delta(\alpha) > \frac{\pi}{2n} \end{cases}$$

with

$$\delta(\alpha) = \left\lfloor \frac{n\alpha}{\pi} \right\rfloor \frac{\pi}{n}.$$

$p(0, \alpha)$ is strictly decreasing for $0 < \alpha < \frac{\pi}{2n}$, which can be seen as follows: We denote by $f_0^*(\alpha)$ the restriction of $f_0(\alpha)$ to the intervall $[0, \frac{\pi}{n})$. It may be written as

$$f_0^*(\alpha) = 2 \cos^2 \frac{\pi}{2n} \left[\sin \alpha + \sin \left(\frac{\pi}{n} - \alpha \right) + \alpha \cos \left(\frac{\pi}{n} - \alpha \right) + \left(\frac{\pi}{n} - \alpha \right) \cos \alpha \right].$$

One finds

$$f_0^{*'}(\alpha) = \frac{d}{d\alpha} f_0^*(\alpha) = 2 \cos^2 \frac{\pi}{2n} \left[\alpha \sin \left(\frac{\pi}{n} - \alpha \right) - \left(\frac{\pi}{n} - \alpha \right) \sin \alpha \right]$$

and hence

$$\frac{f_0^{*'}(\alpha)}{\sin \alpha \sin \left(\frac{\pi}{n} - \alpha \right)} = 2 \cos^2 \frac{\pi}{2n} \left(\frac{\alpha}{\sin \alpha} - \frac{\frac{\pi}{n} - \alpha}{\sin \left(\frac{\pi}{n} - \alpha \right)} \right).$$

For $0 < \alpha \leq \frac{\pi}{2n}$, we have $\alpha \leq \frac{\pi}{n} - \alpha$, and therefore,

$$\frac{\alpha}{\sin \alpha} - \frac{\frac{\pi}{n} - \alpha}{\sin \left(\frac{\pi}{n} - \alpha \right)} \leq 0.$$

It follows that $f_0^{*'}(\alpha) \leq 0$ and hence $p(0, \alpha) \leq 0$ in $0 < \alpha \leq \frac{\pi}{2n}$, where the equality signs hold only if $\alpha = \frac{\pi}{2n}$. Due to the symmetry of $p(0, \alpha)$ (see Theorem 2), $p(0, \alpha)$ is strictly increasing in $\frac{\pi}{2n} < \alpha < \frac{\pi}{n}$. Therefore, the probability of at least one intersection is strictly increasing in $0 < \alpha < \frac{\pi}{2n}$ and strictly decreasing in $\frac{\pi}{2n} < \alpha < \frac{\pi}{n}$ (cp. [4]).

Due to its additivity, the expectation $\sum_{i=0}^{2M} i p(i, \alpha)$ of the number of intersections is always given by $2n(\lambda + \mu)/\pi$.

3 Special cases

The probability of at least one intersection is given by

$$\frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda\mu}{\pi} f_0(\alpha).$$

This is one result of Theorem 2.1 in [4].

For $\mu = 0$ one gets the result for one lattice \mathcal{R}_a of parallel lines in [2, pp. 17-18].

Fig. 4, ..., 9 show diagrams with the intersection probabilities $p(i, \alpha)$, $0 \leq \alpha \leq \pi/n$, for $\lambda = 1/3$, $\mu = 1/4$ and $n = 5$.

Using the formulas in Theorem 1, we get the following approximate expressions in the case $n = 5$:

$$\begin{aligned} p(0, \alpha) &\approx 1 - 0,87098(\lambda + \mu) + c_0\lambda\mu \\ p(1, \alpha) &\approx 0,71465(\lambda + \mu) - c_1\lambda\mu \\ p(2, \alpha) &\approx 1,00054(\lambda + \mu) - c_2\lambda\mu \\ p(3, \alpha) &\approx 0,155792(\lambda + \mu) + c_3\lambda\mu \\ p(i, \alpha) &\approx c_i\lambda\mu, \quad i = 4, 5, 6, \end{aligned}$$

with

$$\begin{aligned} c_0 &= 3,50133, \quad c_1 = 2,67478, \quad c_2 = 3,23888, \quad c_3 = 0,854102, \\ c_4 &= 1,23316, \quad c_5 = 0,292814, \quad c_6 = 0,0322554 \end{aligned}$$

if $\alpha = 0, \pi/5, 2\pi/5$, and

$$\begin{aligned} c_0 &= 3,49988, \quad c_1 = 2,67367, \quad c_2 = 3,22768, \quad c_3 = 0,840122, \\ c_4 &= 1,21437, \quad c_5 = 0,330696, \quad c_6 = 0,0162876 \end{aligned}$$

if $\alpha = \pi/10, 3\pi/10, \pi/2$.

From the calculation of many special cases, we conjecture that $p(i, \alpha)$ is strictly increasing in $0 < \alpha < \frac{\pi}{2n}$ if $i \in \{1, \dots, M-1\}$ or $i = 2M-1$, and strictly decreasing in this intervall if $i \in \{M, \dots, 2M-2\}$ or $i = 2M$.

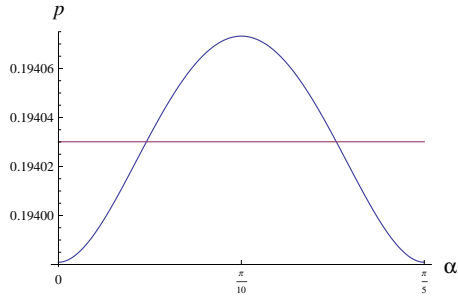


Figure 4: $p(1, \alpha)$

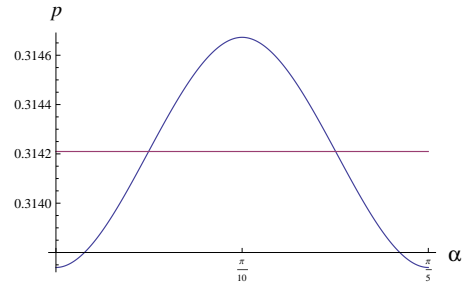


Figure 5: $p(2, \alpha)$

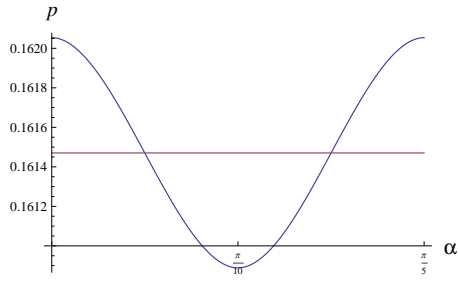


Figure 6: $p(3, \alpha)$

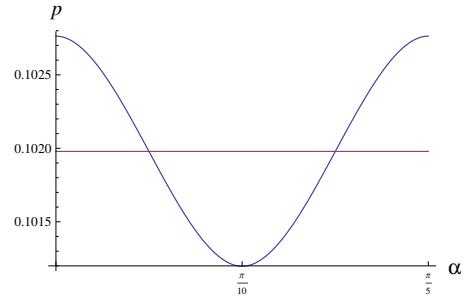


Figure 7: $p(4, \alpha)$

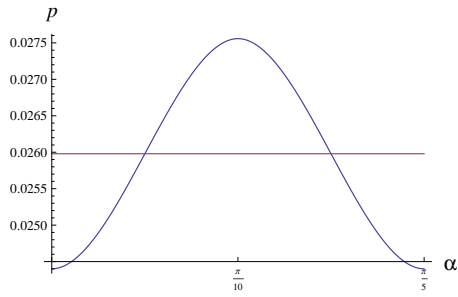


Figure 8: $p(5, \alpha)$

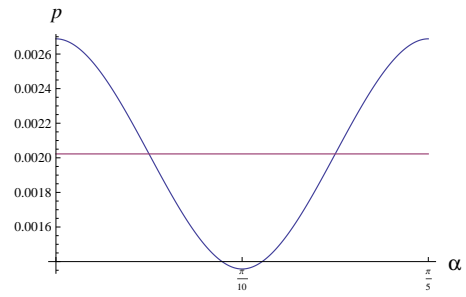


Figure 9: $p(6, \alpha)$

4 Distribution functions

In the following, let $X_{n,\alpha}$ denote the ratio

$$\frac{\text{number of intersections between } \mathcal{S}_{n,\ell} \text{ and } \mathcal{R}_{a,b,\alpha}}{n}$$

(short: *relative number of intersections*) and $F_{n,\alpha} : \mathbb{R} \rightarrow [0, 1]$ the distribution function of $X_{n,\alpha}$,

$$F_{n,\alpha}(\xi) = P(X_{n,\alpha} \leq \xi) = \begin{cases} 0 & \text{for } -\infty < \xi < 0, \\ \sum_{i=0}^{\lfloor n\xi \rfloor} p(i, \alpha) & \text{for } 0 \leq \xi < \frac{2M}{n}, \\ 1 & \text{for } \frac{2M}{n} \leq \xi < \infty. \end{cases}$$

We put

$$X_{n,\lambda} := \frac{\text{number of intersections between } \mathcal{S}_{n,\ell} \text{ and } \mathcal{R}_a}{n} \quad \text{and} \\ X_{n,\mu} := \frac{\text{number of intersections between } \mathcal{S}_{n,\ell} \text{ and } \mathcal{R}_b}{n}.$$

In the case of the independence of $X_{n,\lambda}$ and $X_{n,\mu}$, the distribution function F_n of $X_n := X_{n,\lambda} + X_{n,\mu}$ is given by

$$F_n(\xi) = P(X_n \leq \xi) = \begin{cases} 0 & \text{for } -\infty < \xi < 0, \\ \sum_{i=0}^{\lfloor n\xi \rfloor} \sum_{k=0}^i p_\lambda(k) p_\mu(i-k) & \text{for } 0 \leq \xi < \frac{2M}{n}, \\ 1 & \text{for } \frac{2M}{n} \leq \xi < \infty, \end{cases}$$

where

$$p_\lambda(i) := \begin{cases} p(i, \alpha), & \text{if } 0 \leq i \leq M, \\ 0, & \text{if } M+1 \leq i \leq 2M, \end{cases}$$

if $\mu = 0$ and $\lambda \neq 0$, and

$$p_\mu(i) := \begin{cases} p(i, \alpha), & \text{if } 0 \leq i \leq M, \\ 0, & \text{if } M+1 \leq i \leq 2M, \end{cases}$$

if $\lambda = 0$ and $\mu \neq 0$.

The horizontal lines in the diagrams in Fig. 4, ..., 9 show the values of the probabilities

$$p^*(i) = \sum_{k=0}^i p_\lambda(k) p_\mu(i-k).$$

The question arise if an angle α exist such that $F_n \equiv F_{n,\alpha}$. The calculation of many examples shows that it is (in general) not possible to find such a value of α for finite n . Therefore, $X_{n,\lambda}$ and $X_{n,\mu}$ are (in general) dependent random variables.

The random variables $X_{n,\lambda}$ and $X_{n,\mu}$ converge uniformly to the random variables X_λ with distribution function

$$F_\lambda(\xi) = \begin{cases} 0 & \text{for } -\infty < \xi < 0, \\ 1 - 2\lambda \cos \pi \xi & \text{for } 0 \leq \xi < \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \leq \xi < \infty, \end{cases}$$

and X_μ with distribution function

$$F_\mu(\xi) = \begin{cases} 0 & \text{for } -\infty < \xi < 0, \\ 1 - 2\mu \cos \pi \xi & \text{for } 0 \leq \xi < \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \leq \xi < \infty, \end{cases}$$

respectively (see [2, p. 24]). If X_λ and X_μ are independent, the distribution of $X_\lambda + X_\mu$ can be calculated with the convolution

$$F(\xi) = P(X_\lambda + X_\mu \leq \xi) = \int_{-\infty}^{\infty} F_\lambda(\xi - \eta) dF_\mu(\eta) \quad \text{see [6, p. 90]},$$

which yields

$$F(\xi) = \begin{cases} 0 & \text{for } -\infty < \xi < 0, \\ 1 - 2(\lambda + \mu) \cos \pi \xi & \text{for } 0 \leq \xi < \frac{1}{2}, \\ -2\lambda\mu(\pi \xi \sin \pi \xi - 2 \cos \pi \xi) & \text{for } \frac{1}{2} \leq \xi < 1, \\ 1 - 2\lambda\mu \pi (1 - \xi) \sin \pi \xi & \text{for } 1 \leq \xi < \infty, \\ 1 & \text{for } 1 \leq \xi < \infty, \end{cases} \quad (1)$$

(cf. [5]). The following theorem shows that F is not only the distribution function of the sum $X_\lambda + X_\mu$ but also of the random variable $X := \lim_{n \rightarrow \infty} X_{n,\alpha}$. Therefore, $X_{n,\lambda}$ and $X_{n,\mu}$ are asymptotically independent.

Theorem 3. *As $n \rightarrow \infty$, the random variables $X_{n,\alpha}$ converge to the random variable X whose distribution function is given by formula (1).*

Proof. For fixed coordinates (x, y) of the centre point of $\mathcal{S}_{n,\ell}$, the relative number of intersections tends to $\xi = (\sigma + \tau)/(2\pi)$ as $n \rightarrow \infty$ (see Fig. 10). The outer parallelogram is the fundamental cell \mathcal{F} . $\sigma = \sigma(x, y)$ is the angle of possible intersections with \mathcal{R}_a , and $\tau = \tau(x, y)$ the angle of possible intersections with \mathcal{R}_b .

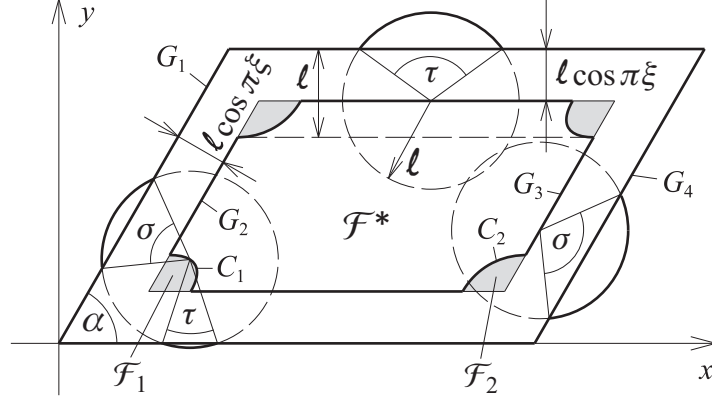


Figure 10: Calculation of F for $0 \leq \xi < \frac{1}{2}$

At first we consider the situation for fixed value of ξ with $0 \leq \xi < 1/2$. The relative number of intersections is equal to ξ if the centre point of $\mathcal{S}_{n,\ell}$ with $n \rightarrow \infty$ lies on the boundary curve of the set $\mathcal{F}^* \subset \mathcal{F}$; it is $< \xi$ if the centre point lies inside \mathcal{F}^* . (\mathcal{F}^* is the inner parallelogram without the four grey coloured sets in its corners.) We denote by A_1 and A_2 the areas of \mathcal{F}_1 and \mathcal{F}_2 respectively. Therefore, the limit distribution is given by

$$\begin{aligned} F(\xi) &= \frac{\text{Area } \mathcal{F}^*(\xi)}{\text{Area } \mathcal{F}} = \frac{(a - 2\ell \cos \pi\xi)(b - 2\ell \cos \pi\xi)/\sin \alpha - 2(A_1 + A_2)}{ab/\sin \alpha} \\ &= \frac{[ab - 2\ell(a + b) \cos \pi\xi - 4\ell^2 \cos^2 \pi\xi] - 2(A_1 + A_2) \sin \alpha}{ab} \\ &= 1 - 2(\lambda + \mu) \cos \pi\xi + 4\lambda\mu \cos^2 \pi\xi - \frac{2(A_1 + A_2) \sin \alpha}{ab}. \end{aligned}$$

In the following, we need the equations of the lines G_1, \dots, G_4 . They are respectively defined in Hesse normal form by

$$\begin{aligned} G_1 &= \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha = 0\}, \\ G_2 &= \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha - \ell \cos \pi\xi = 0\}, \\ G_3 &= \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha - (a - \ell \cos \pi\xi) = 0\}, \\ G_4 &= \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha - a = 0\}. \end{aligned}$$

The subset $\mathcal{F}_1 \subset \mathcal{F}$ is given by

$$\mathcal{F}_1 = \{(x, y) \in \mathbb{R}^2 \mid \ell \cos \pi\xi \leq y \leq \ell, \ g_2(y) \leq x \leq f_1(y)\},$$

where

$$g_2(y) = \frac{1}{\sin \alpha} (y \cos \alpha + \ell \cos \pi\xi)$$

is the equation of G_2 , and $f_1(y)$ the equation of the curve C_1 . We get the equation of C_1 from

$$\xi = \frac{\sigma + \tau}{2\pi} = \frac{1}{\pi} \left(\arccos \frac{x \sin \alpha - y \cos \alpha}{\ell} + \arccos \frac{y}{\ell} \right)$$

which yields

$$f_1(y) = \frac{1}{\sin \alpha} \left[\ell \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) + y \cos \alpha \right]; \quad (2)$$

therefore,

$$f_1(y) - g_2(y) = \frac{\ell}{\sin \alpha} \left[\cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) - \cos \pi \xi \right].$$

So the area of \mathcal{F}_1 is given by

$$\begin{aligned} A_1 &= \int_{\ell \cos \pi \xi}^{\ell} [f_1(y) - g_2(y)] dy \\ &= \frac{\ell}{\sin \alpha} \underbrace{\int_{\ell \cos \pi \xi}^{\ell} \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) dy}_{=: I} - \frac{\ell \cos \pi \xi}{\sin \alpha} \int_{\ell \cos \pi \xi}^{\ell} dy. \end{aligned}$$

We calculate the integral I . With the substitution $u = y/\ell$, one finds

$$\begin{aligned} I &= \ell \int_{\cos \pi \xi}^1 \cos(\pi \xi - \arccos u) du \\ &= \ell \left[\cos \pi \xi \int_{\cos \pi \xi}^1 \cos(\arccos u) du + \sin \pi \xi \int_{\cos \pi \xi}^1 \sin(\arccos u) du \right] \\ &= \ell \left[\cos \pi \xi \int_{\cos \pi \xi}^1 u du + \sin \pi \xi \int_{\cos \pi \xi}^1 \sqrt{1 - u^2} du \right] \\ &= \frac{\ell}{2} \left[u^2 \cos \pi \xi + \left(u \sqrt{1 - u^2} + \arcsin u \right) \sin \pi \xi \right]_{\cos \pi \xi}^1 = \frac{\ell}{2} \pi \xi \sin \pi \xi \end{aligned}$$

and hence

$$A_1 = \frac{\ell^2}{2 \sin \alpha} (\pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi).$$

Now we calculate the area A_2 of

$$\mathcal{F}_2 = \{(x, y) \in \mathbb{R}^2 \mid \ell \cos \pi \xi \leq y \leq \ell, f_2(y) \leq x \leq g_3(y)\},$$

where

$$g_3(y) = \frac{1}{\sin \alpha} (a + y \cos \alpha - \ell \cos \pi \xi)$$

is the equation of the line G_3 , and $f_2(y)$ the equation of the curve C_2 . One gets the equation of C_2 from

$$\xi = \frac{\sigma + \tau}{2\pi} = \frac{1}{\pi} \left(\arccos \frac{-(x \sin \alpha - y \cos \alpha - a)}{\ell} + \arccos \frac{y}{\ell} \right)$$

which gives

$$f_2(y) = \frac{1}{\sin \alpha} \left[a + y \cos \alpha - \ell \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) \right], \quad (3)$$

and hence

$$g_3(y) - f_2(y) = \frac{\ell}{\sin \alpha} \left[\cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) - \cos \pi \xi \right] = f_1(y) - g_2(y).$$

Due to Cavallieri's principle, we have found that $A_2 = A_1$; therefore,

$$\begin{aligned} \frac{2(A_1 + A_2) \sin \alpha}{ab} &= \frac{2\ell^2(\pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi)}{ab} \\ &= 2\lambda\mu(\pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi) \end{aligned}$$

and

$$\begin{aligned} F(\xi) &= 1 - 2(\lambda + \mu) \cos \pi \xi + 4\lambda\mu \cos^2 \pi \xi - 2\lambda\mu (\pi \xi \sin \pi \xi \\ &\quad - 2 \cos \pi \xi + 2 \cos^2 \pi \xi) \\ &= 1 - 2(\lambda + \mu) \cos \pi \xi - 2\lambda\mu (\pi \xi \sin \pi \xi - 2 \cos \pi \xi). \end{aligned}$$

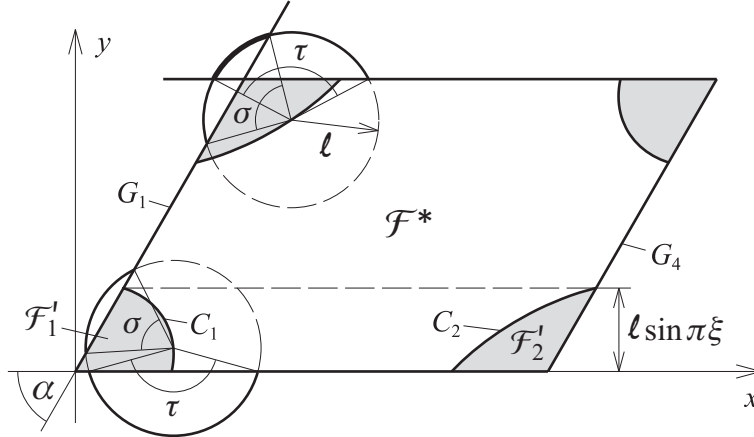


Figure 11: Calculation of F for $\frac{1}{2} \leq \xi < 1$

Now we consider the situation for fixed value of ξ with $\frac{1}{2} \leq \xi < 1$ (Fig. 11). The parallelogram is the fundamental cell \mathcal{F} . \mathcal{F}^* is \mathcal{F} without

the four grey coloured sets in its corners. The limit distribution is given by

$$F(\xi) = \frac{\text{Area } \mathcal{F}^*(\xi)}{\text{Area } \mathcal{F}} = \frac{ab/\sin \alpha - 2(A'_1 + A'_2)}{ab/\sin \alpha} = 1 - \frac{2(A'_1 + A'_2) \sin \alpha}{ab},$$

where A'_1 and A'_2 are the areas of \mathcal{F}'_1 and \mathcal{F}'_2 respectively. The subset $\mathcal{F}'_1 \subset \mathcal{F}$ is defined by

$$\mathcal{F}'_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \ell \sin \pi \xi, \ g_1(y) \leq x \leq f_1(y)\},$$

where $g_1(y) = y \cot \alpha$ is the equation of G_1 , and $f_1(y)$ the equation of C_1 (see (2)). Here the upper limit for the variable y is obtained from

$$\xi = \frac{1}{2} + \frac{\tau}{2\pi} = \frac{1}{2} + \frac{1}{\pi} \arccos \frac{y}{\ell} \implies y = \ell \sin \pi \xi.$$

So we have

$$\begin{aligned} f_1(y) - g_1(y) &= y \cot \alpha + \frac{\ell}{\sin \alpha} \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) - y \cot \alpha \\ &= \frac{\ell}{\sin \alpha} \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) \end{aligned}$$

and

$$A'_1 = \int_0^{\ell \sin \pi \xi} [f_1(y) - g_1(y)] dy = \frac{\ell}{\sin \alpha} \int_0^{\ell \sin \pi \xi} \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) dy.$$

Using the substitution $u = y/\ell$, we get

$$\begin{aligned} A'_1 &= \frac{\ell^2}{\sin \alpha} \int_0^{\sin \pi \xi} \cos(\pi \xi - \arccos u) du \\ &= \frac{\ell^2}{2 \sin \alpha} \left[u^2 \cos \pi \xi + u \sqrt{1 - u^2} \sin \pi \xi + \arcsin u \sin \pi \xi \right]_0^{\sin \pi \xi} \\ &= \frac{\ell^2}{2 \sin \alpha} \left[\sin^2 \pi \xi \cos \pi \xi + \sin \pi \xi \sqrt{\cos^2 \pi \xi} \sin \pi \xi + \arcsin(\sin \pi \xi) \sin \pi \xi \right]. \end{aligned}$$

From $\frac{1}{2} \leq \xi < 1$, it follows that $\cos \pi \xi \leq 0$ and $\arcsin(\sin \pi \xi) = \pi(1 - \xi)$; therefore,

$$\begin{aligned} A'_1 &= \frac{\ell^2}{2 \sin \alpha} \left[-\sin^2 \pi \xi |\cos \pi \xi| + \sin^2 \pi \xi |\cos \pi \xi| + \pi(1 - \xi) \sin \pi \xi \right] \\ &= \frac{\ell^2}{2 \sin \alpha} \pi(1 - \xi) \sin \pi \xi. \end{aligned}$$

The subset $\mathcal{F}'_2 \subset \mathcal{F}$ is defined by

$$\mathcal{F}'_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \ell \sin \pi \xi, \ f_2(y) \leq x \leq g_2(y)\},$$

where

$$g_4(y) = \frac{1}{\sin \alpha} (a + y \cos \alpha)$$

is the equation of G_4 , and $f_2(y)$ the equation of C_2 (see (3)). We get

$$g_4(y) - f_2(y) = \frac{\ell}{\sin \alpha} \cos \left(\pi \xi - \arccos \frac{y}{\ell} \right) = f_1(y) - g_1(y).$$

Due to Cavallieri's principle, we have found that $A'_2 = A'_1$. Therefore,

$$F(\xi) = \frac{ab/\sin \alpha - 4A}{ab/\sin \alpha} = 1 - \frac{4A \sin \alpha}{ab} = 1 - 2\lambda\mu \pi(1 - \xi) \sin \pi \xi,$$

and the proof is complete. \square

Note the interesting fact that the limit distribution F is independent of the angle α ! It is the same limit distribution as for the distribution functions of corresponding *clusters of needles* (with equal values of λ and μ , respectively) [5, p. 221, Theorem 2].

The diagrams in Fig. 12 and Fig. 13 show for $\lambda = 1/3$ and $\mu = 1/4$ examples of distribution functions and the limit distribution F .

The calculation of many special cases show (as the diagrams suggest) that it is most likely that the $F_{n,\alpha}$ converge *uniformly* to F .

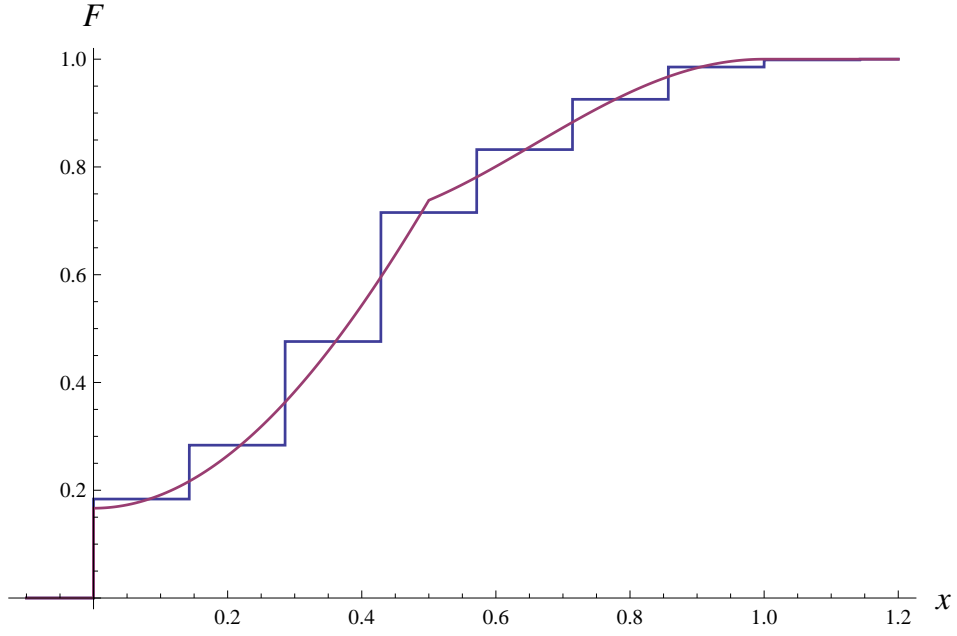


Figure 12: $F_{7,\alpha}$, $\alpha = k\pi/7$, $k = 0, 1, \dots, 3$, and F

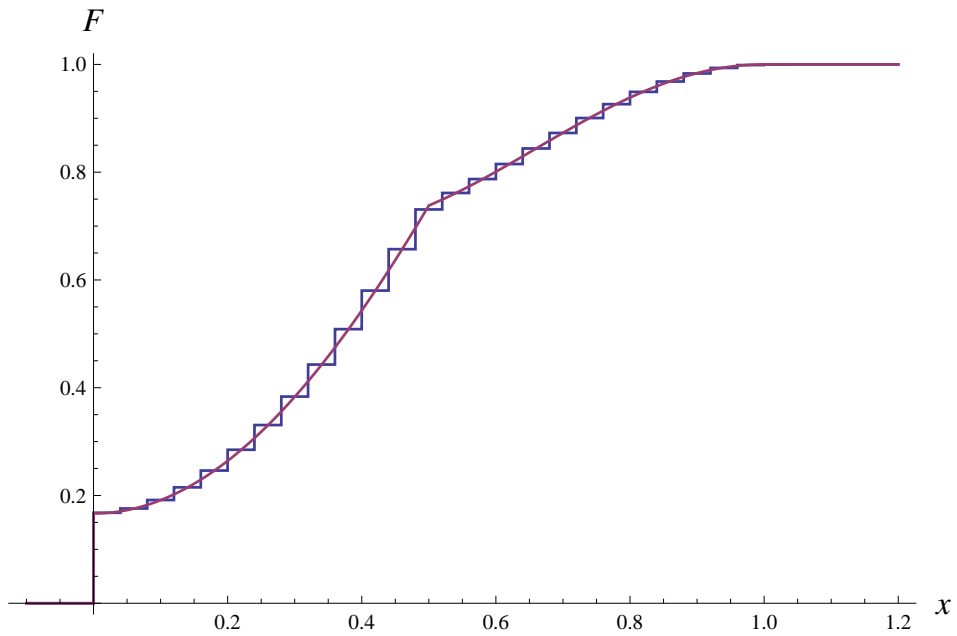


Figure 13: $F_{25,\alpha}$, $\alpha = k\pi/25$, $k = 0, 1, \dots, 12$, and F

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